WEIGHTED PÁL-TYPE INTERPOLATION ON THE UNIT CIRCLE

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Abstract: In this paper, we consider explicit forms and convergence of Pal- type (0;1) interpolation on two disjoint set of nodes, which are obtained by projecting vertically the zeros of the $(1-x^2)P_n^{(\alpha,\beta)}(x)$ and $P_n^{(\alpha,\beta)'}(x)$ respectively on the unit circle, where $P_n^{(\alpha,\beta)}(x)$ stands for Jacobi polynomial.

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I. INTRODUCTION

In a paper, L.G.Pal [10] proved that when function values are prescribed on one set of n points and derivatives values on another set of $n-1$ points, then there exists no unique polynomial of degree $\leq 2n-2$, but by prescribing function value at one more point not belonging to the former set of n points there exists a unique polynomial of degree $\leq 2n-1$. Later, L.Szilli [13] made a similar study on the zeros of $\pi_n(x)$ and $\pi'_n(x) = -n(n-1)P_{n-1}(x)$, $P_n(1)=1$. On the same set of nodes, Akhlaghi and Sharma [1] proved regularity and explicit representation for the (0;2)-case, which means value of function are prescribed on one set and second derivative values on another set.

In 2003, H.P.Dikshit [8] also considered the Pál-type interpolation on non- uniformly distributed nodes on the unit circle. Later, S.Bahadur [2], [3] considered the interscaled set of nodes, which are the zeros of two different polynomial. She considered (0;0,2) and (0,2;0) - interpolation on the unit circle. Further, the authors [5],[6] have dealt with Lagrange – Hermite and Hermite – Lagrange interpolation problems in two different papers taking the two different set of nodes namely the zeros of $(1-x^2)P_n^{(\alpha,\beta)}(x)$ and $P_n^{(\alpha,\beta)'}(x)$ respectively on the unit circle, where $P_n^{(\alpha,\beta)}(x)$ stands for Jacobi polynomial.

In this paper, we consider (0;1) Pál-type interpolation on the vertically projected zeros of $(1-x^2)P_n^{(\alpha,\beta)}(x)$ and $P_n^{(\alpha,\beta)'}(x)$ on the unit circle and establish a convergence theorem for the same. In section 2, we give some preliminaries and in section 3, we describe the problem and obtained the regularity of the same. In section 4, we give the explicit formulae of the interpolatory polynomials. In sections 5 and 6, estimation and convergence of interpolatory polynomials are considered respectively.

II. PRELIMINARIES

In this section, we shall give some well-known results, which we shall use.

The differential equation satisfied by $P_n^{(\alpha,\beta)}(x)$ is

$$(2.1) \quad (1-x^2)P_n^{(\alpha,\beta)''}(x) + [\beta - \alpha - (\alpha + \beta + 2)x]P_n^{(\alpha,\beta)'}(x) + n(n + \alpha + \beta + 1)P_n^{(\alpha,\beta)}(x) = 0$$

$$(2.2) \quad W(z) = \prod_{k=1}^{2n} (z - z_k) = K_n P_n^{(\alpha,\beta)} \left(\frac{1+z^2}{2z} \right) z^n$$

$$(2.3) \quad R(z) = (z^2 - 1)W(z)$$

$$(2.4) \quad H(z) = \prod_{k=1}^{2n-2} (z - t_k) = K_n^* P_n^{(\alpha,\beta)'} \left(\frac{1+z^2}{2z} \right) z^{n-1}$$

We shall require the fundamental polynomials of Lagrange interpolation based on the nodes as zeroes of $R(z)$ and $W(z)$ are given by:

$$(2.5) \quad L_k(z) = \frac{R(z)}{R'(z_k)(z - z_k)}, k = 0(1)2n + 1$$

$$(2.6) \quad l_k(z) = \frac{H(z)}{H'(t_k)(z - t_k)}, k = 1(1)2n - 2$$

We will also use the following results

$$(2.7) \quad (-1)^n W'(z_{n+k}) = W'(z_k) = -\frac{1}{2} K_n P_n^{(\alpha, \beta)'}(x_k) (1 - z_k^2) z_k^{n-2}, k = 1(1)n$$

$$(2.8) \quad (-1)^{n-1} W''(z_{n+k}) = W''(z_k) = -\frac{1}{2} K_n P_n^{(\alpha, \beta)'}(x_k) \left[\begin{array}{c} -\left(\frac{2(\beta - \alpha)z_k}{-(\alpha + \beta + 2)(1 + z_k^2)} \right) \\ + 2n(1 - z_k^2) - 2 \end{array} \right] z_k^{n-3}, k = 1(1)n$$

$$(2.9) \quad R'(z_k) = (z_k^2 - 1)W'(z_k)$$

$$(2.10) \quad R''(z_k) = W'(z_k) [4z_k^2 + 2(\beta - \alpha)z_k - (\alpha + \beta + 2)(1 + z_k^2) - 2n(1 - z_k^2) + 2] z_k^{-1}$$

We will also use the following well known inequalities (see [11])

$$(2.11) \quad (1 - x^2)^{\frac{1}{2}} P_n^{(\alpha, \beta)}(x) = o(n^{\alpha-1}) \text{ for } \alpha > 0, x \in [-1, 1]$$

$$(2.12) \quad (1 - x_k^2)^{-1} \sim \left(\frac{k}{n} \right)^{-2}$$

$$(2.13) \quad \left| P_n^{(\alpha, \beta)'}(x_k) \right| \sim k^{-\alpha - \frac{3}{2}} n^{\alpha+2}$$

$$(2.14) \quad \left| P_n^{(\alpha, \beta)}(x) \right| = o(n^\alpha),$$

$$(2.15) \quad \left| P_n^{(\alpha, \beta)'}(x) \right| = o(n^{\alpha+2})$$

$$(2.16) \quad \left| P_n^{(\alpha, \beta)}(x_k) \right| \sim k^{-\alpha - \frac{1}{2}} n^\alpha$$

$$(2.17) \quad \left| P_n^{(\alpha, \beta)''}(x) \right| = o(n^{\alpha+4})$$

$$(2.18) \quad \left| P_n^{(\alpha, \beta)''}(x_k) \right| \sim k^{-\alpha - \frac{5}{2}} n^{\alpha+4}$$

III. The Problem and Regularity

Let

$$Z_n = \{z_0 = 1, z_{2n+1} = -1,$$

$$z_k = \cos \theta_k + i \sin \theta_k, z_{n+k} = -z_k, k = 1(1)n\}$$

$$T_n = \{t_0 = 1, t_{2n-1} = -1,$$

$$t_k = \cos \phi_k + i \sin \phi_k, t_{n+k} = -t_k, k = 1(1)n-1\}$$

be the vertically projected zeros of the $(1 - x^2)P_n^{(\alpha, \beta)}(x)$ and $P_n^{(\alpha, \beta)'}(x)$ respectively on the unit circle, where

$P_n^{(\alpha, \beta)}(x)$ stands for Jacobi polynomial such that $1 > x_1 > x_2 > \dots > x_n > -1$. In this paper, we seek to

determine a polynomial $R_n(z)$ of lowest possible degree such that

$$(3.1) \quad \begin{cases} R_n(z_k) = \alpha_k, k = 0(1)2n+1 \\ \left[w(z) R_n(z) \right]'_{z=t_k} = \beta_k, k = 1(1)2n-2, w(z) = z^{-n} \end{cases}$$

where α_k and β_k are arbitrary given complex numbers. We shall also establish the convergence theorem for $R_n(z)$.

Theorem 1: $R_n(z)$ is regular on Z_n and T_n

Proof: It is sufficient, if we show, the unique solution of (3.1) is $R_n(z) \equiv 0$, when all data $\alpha_k = \beta_k = 0$. In this case, we have $R_n(z) = W(z)q(z)$,

where $q(z)$ is a polynomial of degree $\leq 2n-1$. Obviously $R_n(z_k) = 0$ for $k = 0(1)2n+1$.

$$\text{From, } [W(z)R_n(z)]_{z=t_k} = [P_n^{(\alpha, \beta)}(x) q(z)]'_{z=t_k}$$

$$\text{we get, } P_n^{(\alpha, \beta)}(t_k) q'(t_k) = 0$$

Thus, we can write

$$q(z) = a \int_{-1}^z H(z) dz$$

In addition

$$q(\pm 1) = 0, \text{ we get } R_n(z) \equiv 0$$

Hence, the theorem follows.

IV. Explicit Representation of Interpolatory Polynomials

We shall write $R_n(z)$ satisfying (3.2) as:

$$(4.1) \quad R_n(z) = \sum_{k=0}^{2n+1} \alpha_k A_k(z) + \sum_{k=1}^{2n-2} \beta_k B_k(z)$$

where $A_k(z)$ and $B_k(z)$ are unique polynomials, each of degree at most $4n-1$ satisfying the conditions:

$$(4.2) \quad \begin{cases} A_k(z_j) = \delta_{jk}, \quad j, k = 0(1)2n+1 \\ [z^{-n} A_k(z)]'_{z_j} = 0, \quad j = 1(1)2n-2, k = 1(1)2n \end{cases}$$

$$(4.3) \quad \begin{cases} B_k(z_j) = 0, \quad j = 0(1)2n+1, k = 1(1)2n-2 \\ [z^{-n} B_k(z)]'_{z_j} = \delta_{jk}, \quad j, k = 1(1)2n-2 \end{cases}$$

THEOREM 2: For $k=1(1)2n-2$, we have,

$$(4.4) \quad B_k(z) = \frac{W(z)}{t_k^{-n+1}} \{J_k(z) + b_k J_{1k}(z)\}$$

where,

$$(4.5) \quad J_k(z) = \int_{-1}^z t l_k(t) dt$$

$$(4.6) \quad J_{1k}(z) = \int_{-1}^z H(t) dt \quad \text{and}$$

$$(4.7) \quad b_k = -\frac{\int_{-1}^1 t l_k(t) dt}{\int_{-1}^1 H(t) dt}$$

PROOF: Using $B_k(1) = 0$ in (4.4), we get (4.7).

THEOREM 3: For $k=1(1)2n$, we have,

$$(4.8) \quad A_k(z) = \frac{H(z) L_k(z)}{H(z_k)} + W(z) [S_k(z) + a_{1k} J_{1k}(z)]$$

where,

$$(4.9) \quad S_k(z) = -\frac{1}{(z_k^2 - 1) H^2(z_k) W'(z_k)} \int_{-1}^z \frac{(z^2 - 1) [H'(z) H(z_k) - H(z) H'(z_k)]}{(z - z_k)} dz$$

where,

$$(4.10) \quad a_{1k} = -\frac{S_k(1)}{J_{1k}(1)}$$

And for $k=0, 2n+1$, we have

$$(4.11) \quad A_0(z) = \frac{W(z) \int_{-1}^z H(t) dt}{W(1) \int_{-1}^1 H(t) dt}$$

$$(4.12) \quad A_{2n+1}(z) = \frac{W(z) \int_{-1}^z H(t) dt}{W(-1) \int_{-1}^1 H(t) dt}$$

V. ESTIMATION OF FUNDAMENTAL POLYNOMIALS

LEMMA 1 [7]: Let $L_k(z)$ be given by (2.5). Then

$$(5.1) \quad \max_{|z|=1} \sum_{k=0}^{2n+1} |L_k(z)| \leq \frac{c}{k^{-\alpha+\frac{3}{2}}}$$

where c is a constant independent of n and z .

LEMMA 2 [5]: Let $l_k(z)$ be given by (2.6). Then

$$(5.2) \quad \max_{|z|=1} \sum_{k=0}^{2n+1} |l_k(z)| \leq \frac{c}{k^{-\alpha-\frac{1}{2}}}$$

where c is a constant independent of n and z .

LEMMA 3: For $|z| \leq 1$, we have

$$(5.3) \quad \sum_{k=1}^{2n-2} |B_k(z)| \leq cn \log n, \quad \alpha < -\frac{1}{2}$$

where $B_k(z)$ is given in theorem 2 and c is a constant independent of n and z .

PROOF: Using Lemma 2, (2.14) and (2.16), we get

$$(5.4) \quad \sum_{k=1}^{2n-2} |B_k(z)| \leq c \frac{n}{k^{-2\alpha}}$$

For $\alpha < -\frac{1}{2}$, (5.4) reduces to (5.3).

LEMMA 4: For $|z| \leq 1$, we have

$$(5.5) \quad \sum_{k=1}^n |A_k(z)| \leq cn^2 \log n, \quad \alpha < -\frac{1}{2}$$

where $A_k(z)$ is given in theorem 3 and c is a constant independent of n and z .

PROOF: Using (2.11) – (2.18) in (4.9), we get

$$(5.6) \quad \sum_{k=1}^n |S_k(z)| \leq c \sum_{k=1}^n \frac{n^{-\alpha+2}}{k^{-2\alpha}}$$

Again, using (2.13), (2.15) and Lemma 1, we get

$$(5.7) \quad \sum_{k=1}^n \left| \frac{H(z) L_k(z)}{H(z_k)} \right| \leq \frac{cn}{k^{-2\alpha-1}}$$

Further, using (2.14), (5.6) and (5.7) in (4.8), we get (5.5).

VI. CONVERGENCE

Let $f(z)$ be analytic for $|z| < 1$ and continuous for $|z| \leq 1$ and $\omega_2(f, \delta)$ be the modulus of continuity of $f(e^{i\theta})$.

THEOREM 4: Let $f(z)$ be continuous in $|z| \leq 1$ and analytic in $|z| < 1$. Let the arbitrary number β_k 's be such that:

$$(6.1) \quad \{\beta_k\} = o(n \omega_2(f, n^{-1})), k = 1(1)2n-2$$

Then $\{R_n\}$ be defined by :

$$(6.2) \quad R_n(z) = \sum_{k=1}^{2n-1} f(z_k) A_k(z) + \sum_{k=1}^{2n-2} \beta_k B_k(z)$$

Satisfies the relation:

$$(6.3) \quad |R_n(z) - f(z)| = o(n^2 \omega_2(f, n^{-1}) \log n), \quad \alpha < -\frac{1}{2},$$

where, $\omega_2(f, n^{-1})$ is the modulus of continuity of $f(z)$.

REMARK 1: Let $f(z)$ be continuous in $|z| \leq 1$ and $f' \in \text{Lip } \nu$, $\nu > 1$, then the sequence $\{R_n\}$ converges uniformly to $f(z)$ in $|z| \leq 1$ follows from (6.3) provided

$$w_2(f, n^{-1}) = o(n^{-1-\nu})$$

To prove theorem 4, we shall need the following:

REMARK 2: Let $f(z)$ be continuous in $|z| \leq 1$ and analytic in $|z| < 1$. Then there exists a polynomial $F_n(z)$ of degree at most $4n - 1$ satisfying Jackson's inequality

$$(6.4) \quad |F_n(z) - f(z)| \leq c w_2(f, n^{-1}), \quad z = e^{i\theta} (0 \leq \theta < 2\pi)$$

And also an inequality due to O.Kiš [9]

$$(6.5) \quad |F_n^{(m)}(z)| \leq c n^m \omega(f, n^{-1}), \quad \text{for } m \in \mathbb{I}^+$$

PROOF: Since $R_n(z)$ be given by (6.2) is a uniquely determined polynomial of degree $\leq 4n - 1$, the polynomial $F_n(z)$ satisfying (6.4) and (6.5) can be expressed as :

$$F_n(z) = \sum_{k=1}^{2n} F_n(z_k) A_k(z) + \sum_{k=1}^{2n-2} F_n'(z_k) B_k(z)$$

$$\begin{aligned} \text{Then,} \quad |R_n(z) - f(z)| &\leq |R_n(z) - F_n(z)| + |F_n(z) - f(z)| \\ &\leq \sum_{k=1}^{2n} |f(z_k) - F_n(z_k)| A_k(z) + \sum_{k=1}^{2n-2} \{|\beta_k| + F_n'(z_k)\} B_k(z) \\ &\quad + |F_n(z) - f(z)| \end{aligned}$$

Using $z = e^{i\theta} (0 < \theta \leq 2\pi)$, (6.1), (6.4), (6.5) and lemmas 3 and 4, we get (6.3).

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